Question 1 **[5, 2 = 7 marks]**

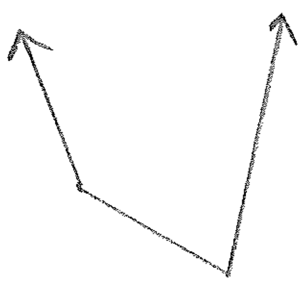
Consider the functions $f(x) = |2x-5|$ *and* $g(x) = |x-4|$.
Let $h(x) = f(x) + 3g(x)$.

a) Write $h(x)$ as a piecewise function free of absolute values.

	$x < \frac{5}{2}$	$\frac{5}{2}$	$\frac{5}{2} < x < 4$	4	$x > 4$
$f(x)$	$-2x+5$		$2x-5$		$2x-5$
$g(x)$	$-x+4$		$-x+4$		$x-4$

$$h(x) = \begin{cases} -5x+17 \checkmark & x \le \frac{5}{2} \\ -x+7 \checkmark & \frac{5}{2} < x \le 4 \\ 5x-17 \checkmark & x > 4 \end{cases}$$

$\left(x = \frac{5}{2} \checkmark, 4 \checkmark\right)$



* If they don't see the 3 g(x)

FT is $h(x) = \begin{cases} 9-3x & x \le \frac{5}{2} \\ x-1 & \frac{5}{2} < x \le 4 \\ 3x+9 & x > 4 \end{cases}$

* only deduct 1m

b) Determine the range of $h(x)$. ✓

min value is $h(4) = 3$

$\therefore \{y : y \ge 3, y \in \mathbb{R}\}$ ✓

(FT is $y \ge \frac{3}{2}$)

Question 2 **[3, 3, 4, 4 = 14 marks]**

a) Determine $\frac{dy}{dx}$ for each of the following. Only minor simplification is required.

(i) $y = x^3 \sin^3(4x)$

$$\frac{dy}{dx} = 3x^2 \sin^3(4x) + x^3 . 3\sin^2(4x)\cos(4x).4 \quad ✓✓$$
$$= 3x^2 \sin^3(4x) + 12x^3 \sin^2(4x)\cos(4x) \quad ✓$$

(ii) $e^y = xy^3$

$$e^y \frac{dy}{dx} = 1 \cdot y^3 + x \cdot 3y^2 \frac{dy}{dx} \quad ✓✓$$

$$\frac{dy}{dx} = \frac{y^3}{e^y - 3xy^2} \quad ✓$$

(iii) $y = \ln\left(\frac{\sqrt{\cos x}}{x^2 \sin x}\right)$

*use log laws

$$y = \ln(\cos x)^{\frac{1}{2}} - \ln(x^2) - \ln(\sin x) \quad ✓$$
$$y = \frac{1}{2}\ln(\cos x) - 2\ln(x) - \ln(\sin x)$$
$$\frac{dy}{dx} = \frac{-\sin x}{2\cos x} - \frac{2}{x} - \frac{\cos x}{\sin x} \quad ✓✓✓$$

b) Determine the exact value of $\lim_{h \to 0} \dfrac{\sin^3\left(\frac{\pi}{6}+h\right)-\frac{1}{8}}{h}$.

$$\left.\frac{d}{dx}\left(\sin^3 x\right)\right|_{x=\frac{\pi}{6}}$$ ✓

$$= \left. 3\sin^2 x \cos x \right|_{x=\frac{\pi}{6}}$$ ✓

$$= 3\left(\frac{1}{2}\right)^2\left(\frac{\sqrt{3}}{2}\right)$$

✓½ ✓½

$$= \frac{3\sqrt{3}}{8}$$ ✓

Question 3 **[2, 1, 2, 2, 1, 3, 4 = 15 marks]**

Consider matrix **T** = $\begin{bmatrix} 4 & 2 \\ -5 & k \end{bmatrix}$

a) In each case, state the value of k which satisfies the given condition.

(i) **T** is singular.

$4k + 10 = 0$ ✓

$\therefore k = \frac{-10}{4}$ ✓

(ii) **T** maps the vertices of a quadrilateral onto a line.

$k = \frac{-10}{4}$ ✓

(iii) **T** maps the triangle with vertices P (0, 0), Q (1, 0), R (0, –1) onto the triangle with vertices P' (0,0) , Q' (4, –5) and R' (–2, –3).

$\begin{pmatrix} 4 & 2 \\ -5 & k \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 4 & -2 \\ -5 & -3 \end{pmatrix}$

$10 - 3k = -1$ ✓

$\therefore k = 3$ ✓

(only 1m if $k = \pm 3$)

(iv) $\mathbf{T}^2 = \begin{bmatrix} 6 & 2 \\ -5 & -1 \end{bmatrix}$

$\begin{pmatrix} 4 & 2 \\ -5 & k \end{pmatrix}\begin{pmatrix} 4 & 2 \\ -5 & k \end{pmatrix}$

$\Rightarrow -20 - 5k = -5$ ✓

$k = -3$ ✓

(v) $\mathbf{T} - 2\mathbf{I} = \begin{bmatrix} 2 & 2 \\ -5 & -1 \end{bmatrix}$, where $\mathbf{I}$ is the 2 × 2 Identity Matrix.

$$T = \begin{pmatrix} 2 & 2 \\ -5 & -1 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$\therefore k = 1 \checkmark$$

(vi) The point (3, 4) is transformed to the point (4, –7) after applying $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, and then applying $\mathbf{T}$.

$$\begin{pmatrix} 4 & 2 \\ -5 & k \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 4 \\ -7 \end{pmatrix}$$

$$\begin{pmatrix} 4 & -2 \\ -5 & -k \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 4 \\ -7 \end{pmatrix} \checkmark$$

$$-15 - 4k = -7 \checkmark$$

$$k = -2 \checkmark$$

b) If $k = 2$ and $\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 0 & 4 \end{bmatrix}$, find $\mathbf{B}$, given that $\mathbf{AB} + \mathbf{B} = \mathbf{T}$.

$$(A+I)B = T$$

$$B = (A+I)^{-1}T \checkmark$$

$$B = \begin{pmatrix} 2 & 3 \\ 0 & 5 \end{pmatrix}^{-1}\begin{pmatrix} 4 & 2 \\ -5 & 2 \end{pmatrix} \checkmark$$

$$= \frac{1}{10}\begin{pmatrix} 5 & -3 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 4 & 2 \\ -5 & 2 \end{pmatrix} \checkmark$$

$$= \begin{pmatrix} \frac{35}{10} & \frac{4}{10} \\ -1 & \frac{4}{10} \end{pmatrix} \checkmark$$

Question 4 **[2, 2, 3 = 7 marks]**

-1 no +c penalised once only on whole exam paper

Determine each of the following indefinite integrals:

a) $\int \frac{3x}{5x^2-2}\,dx$ $= \frac{3}{10} \ln|5x^2-2| + c.$ ✓ ✓

b) $\int 6\sin(\pi - 3t)\,dt$ $= 2\cos(\pi - 3t) + c.$ ✓ ✓

c) $\int \frac{\cos x - \sin x}{(\sin x + \cos x)^2}\,dx$

Let $u = \sin x + \cos x$

$\frac{du}{dx} = \cos x - \sin x$

$\therefore du = (\cos x - \sin x)\,dx$

$= \int \frac{1}{u^2}\,du$ ✓

$= -\frac{1}{u} + c$ ✓

$= \frac{-1}{\sin x + \cos x} + c.$ ✓

Question 5 **[3, 4 = 7 marks]**

A curve is defined parametrically by the equations $x = \frac{1}{e^t}$ and $y = (t+1)^3$.

Find each of the following in terms of t, fully simplifying your answers.

$$\frac{dx}{dt} = \frac{-1}{e^t} \checkmark$$

$$\therefore \frac{dt}{dx} = -e^t$$

a) $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$= 3(t+1)^2 \times -e^t$ ✓

$= -3e^t(t+1)^2$ ✓

b) $\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right) \times \frac{dt}{dx}$ ✓

$= \left[-3e^t(t+1)^2 - 6e^t(t+1)\right] \times -e^t$ ✓✓

$= 3e^{2t}(t+1)(t+1+2)$

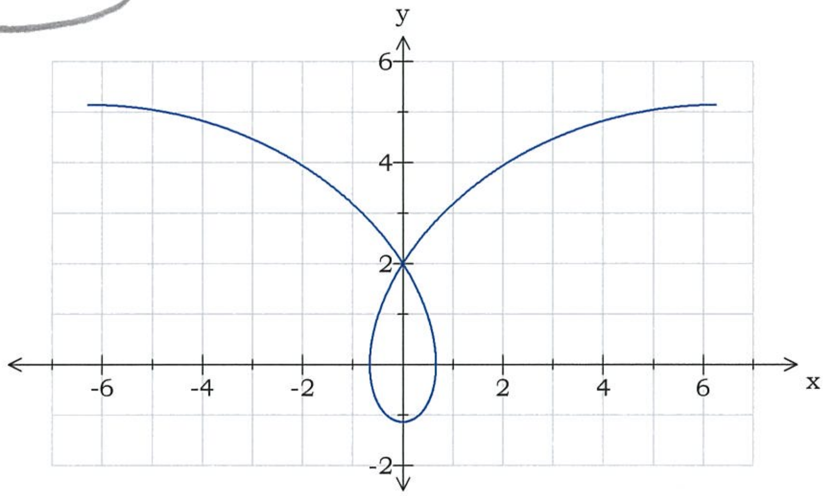
$= 3e^{2t}(t+1)(t+3)$ ✓

Question 6 **[7 marks]**

The *prolate cycloid* represented in the diagram below is defined by the parametric equations

$$x = 2t - \pi \sin t \quad \text{and} \quad y = 2 - \pi \cos t \qquad \text{for } -\pi \le t \le \pi .$$

It crosses itself at the point $(0,2)$. At this point there are two tangent lines.
Find (in exact terms) the equations of each of these lines.



@ $(0,2)$, $2 = 2 - \pi \cos t$

$\therefore \pi \cos t = 0$

$t = \pm \frac{\pi}{2}$ ✓

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$= \pi \sin t \times \frac{1}{2 - \pi \cos t}$ ✓✓

$\left.\frac{dy}{dx}\right|_{t=\frac{\pi}{2}} = \pi(1) \times \frac{1}{2-0}$

$= \frac{\pi}{2}$ ✓

$\therefore$ Eqn is $y = \frac{\pi}{2}x + 2$ ✓

$\left.\frac{dy}{dx}\right|_{t=-\frac{\pi}{2}} = \pi(-1) \times \frac{1}{2-0}$

$= -\frac{\pi}{2}$ ✓

$\therefore$ Eqn is $y = -\frac{\pi}{2}x + 2$ ✓

Question 7 **[6 marks]**

Determine, exactly, the equation of the tangent to the curve defined by $e^{y+3} = ln\,(x-3)$ at the point where $x = e + 3$.

when $x = e+3$, $e^{y+3} = \ln e$
solving gives $y = -3$ ✓

Diff. w.r.t. x :

$$e^{y+3} \cdot \frac{dy}{dx} = \frac{1}{(x-3)}$$ ✓✓

$$\therefore \frac{dy}{dx} = \frac{1}{e^{y+3}(x-3)}$$ ✓

$$\left.\frac{dy}{dx}\right|_{(e+3,-3)} = \frac{1}{e^{-3+3}(e+3-3)}$$

$$= \frac{1}{e}$$ ✓

$\therefore$ Eqn of tangent $\Rightarrow y = \frac{1}{e}x + c$
sub in $(e+3, -3) \Rightarrow c = \frac{-3}{e} - 4$

$\therefore$ Eqn is $y = \frac{1}{e}x - \frac{3}{e} - 4$ ✓

Question 8 **[4, 2, 2 = 8 marks]**

a) Differentiate with respect to x: $y = x^{sin\,(2x)}$

$\ln y = \ln x^{\sin(2x)} = \sin(2x).\ln x$ ✓

Diff. w.r.t. x

$\frac{1}{y}\frac{dy}{dx} = 2\cos(2x)\ln x + \frac{\sin(2x)}{x}$ ✓✓

$\therefore \frac{dy}{dx} = x^{\sin(2x)}\left(2\cos(2x)\ln x + \frac{\sin(2x)}{x}\right)$ ✓

b) Determine each of the following indefinite integrals:

(i) $\int (x^e + e^x)\, dx$

$= \frac{x^{e+1}}{e+1} + e^x + c$ ✓ ✓

-1 if no + c (penalise once only on whole paper)

(ii) $\int (4x+6)(e^{x^2+3x})\, dx$

$= 2e^{x^2+3x} + c$ ✓ ✓

Question 9 **[6, 5 = 11 marks]**

Evaluate each of the following definite integrals, using the suggested substitutions. You must show all of your working.

a) $\displaystyle\int_0^{\frac{\pi}{3}} \frac{\sin^3 x}{\cos^4 x}\, dx$ (see below right) using the substitution $u = \cos x$.

$\frac{du}{dx} = -\sin x$

$\therefore \sin x\, dx = -du$ ✓

$\sin^2 x = 1 - \cos^2 x$

$= 1 - u^2$ ✓

when $x = 0$, $u = \cos 0$

$= 1$. ½

when $x = \frac{\pi}{3}$, $u = \cos\frac{\pi}{3}$

$= \frac{1}{2}$ ½

$\frac{\sin^3 x}{\cos^4 x}$

$= \frac{\sin x \cdot \sin^2 x}{\cos^4 x}$

$= \frac{\sin x(1-\cos^2 x)}{\cos^4 x}$.

$= \displaystyle\int_1^{\frac{1}{2}} \frac{-(1-u^2)}{u^4}\, du$.

$= \displaystyle\int_1^{\frac{1}{2}} \frac{-1}{u^4} + \frac{1}{u^2}\, du$. ✓

$= \left[\frac{1}{3u^3} - \frac{1}{u}\right]_1^{\frac{1}{2}}$. ½ ½

$= \left(\frac{8}{3} - 2\right) - \left(\frac{1}{3} - 1\right)$

$= \frac{4}{3}$. ✓

b) $\int_0^1 \frac{1}{\sqrt{4-x^2}}\,dx$ using the substitution $x = 2\cos\theta$.

$$\frac{dx}{d\theta} = -2\sin\theta$$

$$\therefore dx = -2\sin\theta\, d\theta \quad \checkmark$$

$$\sqrt{4-x^2} = \sqrt{4-4\cos^2\theta}$$

$$= \sqrt{4\sin^2\theta}$$

$$= 2\sin\theta \quad \checkmark$$

when $x=0$, $\theta = \frac{\pi}{2}$ $\frac{1}{2}$

when $x=1$, $\theta = \frac{\pi}{3}$ $\frac{1}{2}$

$$= \int_{\frac{\pi}{2}}^{\frac{\pi}{3}} \frac{1}{2\sin\theta} \times -2\sin\theta\, d\theta$$

$$= \int_{\frac{\pi}{2}}^{\frac{\pi}{3}} -1\, d\theta$$

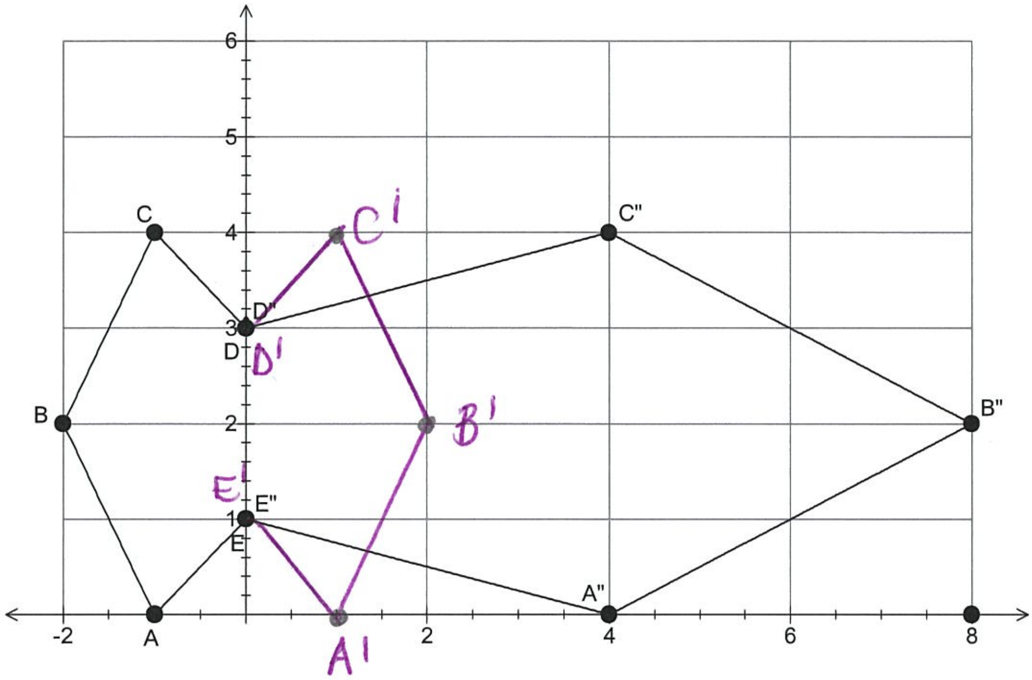
$$= \left[-\theta\right]_{\frac{\pi}{2}}^{\frac{\pi}{3}} \quad \checkmark$$

$$= -\frac{\pi}{3} + \frac{\pi}{2}$$

$$= \frac{\pi}{6} \quad \checkmark$$

Question 10 **[1, 3, 2, 1, 2, 2 = 11 marks]**

The diagram below shows pentagon ABCDE, and the resulting pentagon A"B"C"D"E", after having undergone two matrix transformations, T_1 followed by T_2.



a) Pentagon ABCDE is reflected in the y-axis (T_1). Draw the resulting image A'B'C'D'E' on the diagram above.

b) The pentagon A'B'C'D'E' is then transformed by matrix T_2 to A"B"C"D"E", as shown in the diagram above. Find and describe the transformation represented by matrix T_2.

✓ $\begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$ Horizontal dilation ✓
factor 4 ✓

c) Show how T_1 and T_2 combine to map ABCDE directly to A"B"C"D"E" and give the resulting matrix.

$\begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ ✓ → need to show the order

$= \begin{pmatrix} -4 & 0 \\ 0 & 1 \end{pmatrix}$ ✓

d) Determine the ratio Area ABCDE : Area A"B"C"D"E".

*det of $T_2T_1 = 4$

$1 : 4$.

e) What matrix would transform A"B"C"D"E" back to ABCDE?

$$\begin{pmatrix} -4 & 0 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} -\frac{1}{4} & 0 \\ 0 & 1 \end{pmatrix}$$

✓ ✓

f) Consider $T_3 = \begin{bmatrix} 0 & k \\ k & 0 \end{bmatrix}$. If T_3 is applied to ABCDE, the area of the resulting pentagon will be double the area of ABCDE. Determine the value(s) of k.

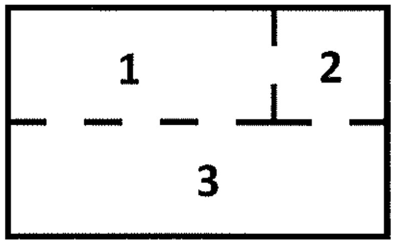
$$|0-k^2| = 2$$

$$\therefore k = \pm\sqrt{2}$$

✓ ✓

Question 11 **[2, 2, 2, 1, 1 = 8 marks]**

A mouse is placed in the maze shown on the right.



During a fixed time interval, the mouse randomly chooses one of the doors (openings) available to it, and moves into the next room. It ***does not*** remain in the room it occupies.

If the mouse started in ***Room 1***, the probabilities that the mouse will be in each room after 1 transition are:

Room 1:	0	(since it must move to another room)
Room 2:	0.25	(since 1 out of the 4 doors available lead to Room 2)
Room 3:	0.75	

a) The first column of the transition matrix is show below. Complete the remainder of the transition matrix.

$$\text{To room}\quad \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \textit{From room} \\ \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 0.5 & 0.75 \\ 0.25 & 0 & 0.25 \\ 0.75 & 0.5 & 0 \end{bmatrix}\end{array}$$

$-\frac{1}{2}$ / ERROR

b) If the mouse starts in Room 3, what is the probability that it is in Room 1 after 4 transitions?

FIND $T^3 \Rightarrow a_{13} \approx 0.61$. (2dp) (see underlined info above)

c) Using appropriate rounding, determine the "**Stable State Matrix**".

$$\begin{pmatrix} 0.4 & 0.4 & 0.4 \\ 0.2 & 0.2 & 0.2 \\ 0.4 & 0.4 & 0.4 \end{pmatrix}$$

d) As the number of transitions becomes large, what is the probability that the mouse will be in Room 2?

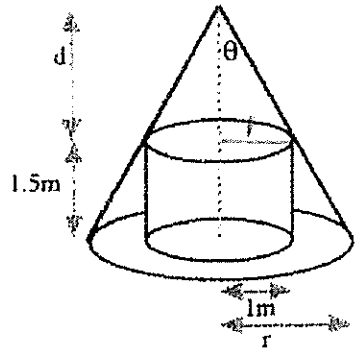
0.2

e) In the long run, what percentage of the time will the mouse spend in Rooms 1 or 2?

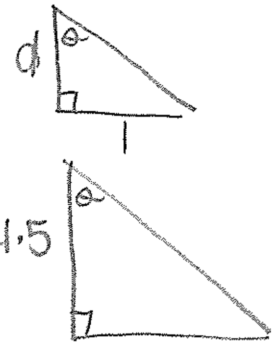
≈ 60%

Question 12 **[2, 2, 4 = 8 marks]**

A cylindrical platform of height 1.5 metres and base radius 1 metre sits on a stage with a vertical axis. To experiment with lighting effects, a point source of light casting a shadow of the cylinder on the stage, is moved upwards so that the angle θ shown in the diagram decreases at a rate of 0.1 radians per second.



a) Show that $d = \dfrac{1.5}{r-1}$.



using similar Δs

$\dfrac{d+1.5}{r} = \dfrac{d}{1}$ ✓

$\therefore d+1.5 = dr$

$1.5 = dr - d$

$1.5 = d(r-1)$ ✓ $\therefore d = \dfrac{1.5}{r-1}$ as required

b) Express r in terms of θ.

$\tan\theta = \dfrac{1}{d}$

$\therefore \tan\theta = \dfrac{r-1}{1.5}$ ✓ $\therefore r = 1.5\tan\theta + 1$ ✓

c) Find the rate at which the radius of the shadow is decreasing when $\theta = \frac{\pi}{6}$.

$\dfrac{dr}{dt} = \dfrac{d\theta}{dt} \times \dfrac{dr}{d\theta}$

$= -0.1 \times \dfrac{1.5}{\cos^2\theta}$ ✓✓

$\left.\dfrac{dr}{dt}\right|_{\theta=\frac{\pi}{6}} = -0.2$ ✓

$\therefore$ decreasing @ 0.2 m/s ✓

(read carefully)

Question 13 **[3, 2, 4 = 9 marks]**

a) Consider the following information about matrices ***A***, ***B*** and ***C*** :

- They are all 2 x 2 matrices
- Matrix ***B*** is a non-singular matrix
- $\boldsymbol{A} = \boldsymbol{BCB}^{-1}$

Find a simplified expression for $\boldsymbol{A}^2$ and $\boldsymbol{A}^3$. Use your results to deduce $\boldsymbol{A}^n$.

$$A^2 = BCB^{-1} \times BCB^{-1}$$
$$= BC^2B^{-1} \quad \checkmark$$
$$A^3 = BC^2B^{-1} \times BCB^{-1}$$
$$= BC^3B^{-1} \quad \checkmark$$

$\therefore$ Deduce that

$$A^n = BC^nB^{-1} \quad \checkmark$$

b) Let $\mathbf{A}^2 - 6\mathbf{A} + 5\mathbf{I} = \mathbf{O}$ where $\mathbf{O} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

(i) Show that $\mathbf{I} = \frac{1}{5}\mathbf{A}(6\mathbf{I} - \mathbf{A})$.

$$A^2 - 6A + 5I = O \quad \therefore \; 5I = 6A - A^2 \quad \checkmark$$
$$= A6I - A^2$$
$$= A(6I - A)$$
$$\therefore I = \frac{1}{5}A(6I - A) \quad \checkmark$$

(ii) Express $\mathbf{A}^4$ in the form $p\mathbf{A} + q\mathbf{I}$.

$A^2 - 6A + 5I = 0$

$\therefore A^2 = 6A - 5I$.

$A^4 = (6A - 5I)(6A - 5I)$ ✓

$= 36A^2 - 30AI - 30AI + 25I^2$ ✓

$= 36A^2 - 60A + 25I$

$= 36(6A - 5I) - 60A + 25I$ ✓

$= 216A - 180I - 60A + 25I$

$= 156A - 155I$. ✓

Question 14 **[1, 3, 1, 3 = 8 marks]**

a) Let $\boldsymbol{A} = \begin{bmatrix} -4 & 20 & 2 \\ -3 & 15 & -3 \\ 7 & -17 & 1 \end{bmatrix}$ and $\boldsymbol{B} = \begin{bmatrix} 2 & 3 & 5 \\ 1 & 1 & 1 \\ 3 & -4 & 0 \end{bmatrix}$.

Determine $\boldsymbol{C} = \boldsymbol{AB}$.

$$C = \begin{pmatrix} 18 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 18 \end{pmatrix}$$

b) Tickets to a concert cost \$2 for children, \$3 for teenagers and \$5 for adults. 570 people attended the concert and the total ticket receipts were \$1950. The ratio of teenagers to children attending was 3 to 4.

(i) Write 3 equations to represent this information, using the variables *c*, *t* and *a* to represent the number of children, teenagers and adults respectively.

✓ $2c + 3t + 5a = 1950$

✓ $c + t + a = 570.$

✓ $3c = 4t.$ (OR $3c - 4t = 0$)

(ii) Represent your answer to b) (i) to write a matrix equation in the form $\boldsymbol{PX} = \boldsymbol{Q}$, where $\boldsymbol{X} = \begin{bmatrix} c \\ t \\ a \end{bmatrix}$.

$$\begin{pmatrix} 2 & 3 & 5 \\ 1 & 1 & 1 \\ 3 & -4 & 0 \end{pmatrix} \begin{pmatrix} c \\ t \\ a \end{pmatrix} = \begin{pmatrix} 1950 \\ 570 \\ 0 \end{pmatrix}$$

(iii) **Use your answer from a)** to determine how many children, teenagers and adults attended the concert.

must use this or only 1m for answer

$$\begin{pmatrix} c \\ t \\ a \end{pmatrix} = \frac{1}{18} A \begin{pmatrix} 1950 \\ 570 \\ 0 \end{pmatrix} \checkmark\checkmark$$

$\therefore$ there were 200 children
150 teenagers
220 adults $\checkmark$

Question 15 **[3, 4, 4 = 11 marks]**

A rocket ship leaves space station A, which is located at $\begin{pmatrix}-20\\40\\20\end{pmatrix}$ km, at 9 am. It travels with a constant velocity of $\begin{pmatrix}60\\120\\360\end{pmatrix}$ km h^{-1}. At some time it is supposed to reach the neighbouring space station B, which is located at $\begin{pmatrix}80\\160\\2020\end{pmatrix}$ km.

a) Show that this rocket ship does not reach space station B.

$\underset{\sim}{r}_A = \langle -20, 40, 20\rangle + t\langle 60, 120, 360\rangle$ km ✓

to reach space station.

$\underset{\sim}{r}_A = \langle 80, 160, 2020\rangle$ ✓

$-20 + 60t = 80$ $\quad$ $40 + 120t = 160$

$t = \frac{5}{3}$ ½ $\quad$ $t = 1$ ½

∴ Does not reach as t values are unequal

b) Find the closest distance between the rocket ship and space station B and the time when this occurs. Answer correct to the nearest minute.

closest distance when ${}_A\underset{\sim}{r}_B \cdot {}_A\underset{\sim}{v}_B = 0$

$$\begin{pmatrix}-20+60t-80\\40+120t-160\\20+360t-2020\end{pmatrix} \cdot \begin{pmatrix}60\\120\\360\end{pmatrix} = 0$$ ✓

ie $t \approx 5.02$ (hours after 9am) ✓ ie @ 2.01pm ✓

$$\text{min dist} = \left|\begin{pmatrix}-20+60t-80\\40+120t-160\\20+360t-2020\end{pmatrix}\right|_{t=5.02} \approx 557\text{km}$$ ✓

A second rocket ship is launched from space station B at 9 am with constant velocity and is aimed to collide with the first rocket at exactly 1 pm.

c) Determine the velocity of the second rocket ship that will ensure collision takes place at the required time.

$\underset{\sim}{r}_2 = \langle 80, 160, 2020 \rangle + t(\underset{\sim}{v}_2)$ ✓

(where $\underset{\sim}{v}_2$ is the velocity of the second rocket)

For collision @ 1pm,

✓✓ $\langle 80, 160, 2020 \rangle + 4\underset{\sim}{v}_2 = \langle -20, 40, 20 \rangle + 4\langle 60, 120, 360 \rangle$

$\therefore \underset{\sim}{v}_2 = \langle 35, 90, -140 \rangle$ km/h ✓

Question 16 **[1, 1, 2, 2 = 6 marks]**

The table below shows the details of a population of kangaroos in a region of Western Australia in 2000.

(*) 1 gen = 2 yrs.

Age (years)	0 – 2	2 – 4	4 – 6	6 - 8	8 - 10
Initial population	1200	1400	1600	810	425
Breeding Rate	0	0.1	3.5	2.5	0.5
Survival Rate	0.4	0.5	0.7	0.2	0

a) Write down the Leslie matrix, L, for this population.

$$L = \begin{pmatrix} 0 & 0.1 & 3.5 & 2.5 & 0.5 \\ 0.4 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0.7 & 0 & 0 \\ 0 & 0 & 0 & 0.2 & 0 \end{pmatrix}$$

$-\frac{1}{2}$ / error

b) What was the expected total population in 2006?

Expect 3 generations in 6 year ✓.

$$\therefore \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \end{pmatrix} L^3 \begin{pmatrix} 1200 \\ 1400 \\ 1600 \\ 810 \\ 425 \end{pmatrix} = (6509.2) \quad \therefore \text{Approx } 6509$$ ✓

c) Find the percentage growth rate between the 3rd and 4th generation.

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \end{pmatrix} L^4 \begin{pmatrix} 1200 \\ 1400 \\ 1600 \\ 810 \\ 425 \end{pmatrix} = (9493.1)$$ ✓

$$\% \text{ change} = \frac{(9493 - 6509)}{6509} \times 100.$$

$$\approx 45.8\%$$ ✓

Due to complaints from local property owners, a decision was made to control the population by culling.

d) To reduce the population growth, 10 % of kangaroos aged between 4 and 8 years are culled at the beginning of every second year. What will the expected population be in 2016?

Note: these also ↓ by 10%.

$$L = \begin{pmatrix} 0 & 0.1 & 3.15 & 2.25 & 0.5 \\ 0.4 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0.63 & 0 & 0 \\ 0 & 0 & 0 & 0.18 & 0 \end{pmatrix}$$

✓ ($-\frac{1}{2}$ error)

In 2016 expect 8 generations

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \end{pmatrix} L^8 \begin{pmatrix} 1200 \\ 1400 \\ 1600 \\ 810 \\ 425 \end{pmatrix} = (7730.6)$$

∴ APPROX 7731. ✓

Question 17 **[2, 3, 2 = 7 marks]**

The lines l_1 and l_2 have equations

$\mathbf{r}_1 = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$ and $\mathbf{r}_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$ respectively, where λ and μ are parameters.

a) Find the acute angle between l_2 and the line joining the points $P(1, -1, 1)$ and $Q(2, -1, -4)$, giving your answer correct to the nearest degree.

$\overrightarrow{PQ} = \langle 1, 0, -5 \rangle$ ✓

$\therefore$ angle between $\langle 2, 1, -1 \rangle$ & $\langle 1, 0, -5 \rangle$

$\approx 56°$ ✓

b) Determine the position vector of the point R that lies on the line joining $P(1, -1, 1)$ and $Q(2, -1, -4)$ such that PR : RQ = 1 : 2.

$\overrightarrow{PR} = \frac{1}{3}\overrightarrow{PQ}$ ✓

$\overrightarrow{OR} = \overrightarrow{OP} + \overrightarrow{PR}$

$= \langle 1, -1, 1 \rangle + \frac{1}{3}\langle 1, 0, -5 \rangle$ ✓

$= \langle \frac{4}{3}, -1, -\frac{2}{3} \rangle$ ✓

c) Find an equation in the form $\mathbf{r} \cdot \mathbf{n} = \rho$, of the plane that passes through $Q(2, -1, -4)$ and is perpendicular to l_1.

plane is $\perp$ to l_1

$\therefore \langle 1, 2, 4 \rangle$ is normal to the plane ✓

$\underset{\sim}{r} \cdot \langle 1, 2, 4 \rangle = \langle 2, -1, -4 \rangle \cdot \langle 1, 2, 4 \rangle$

$\therefore \underset{\sim}{r} \cdot \langle 1, 2, 4 \rangle = -16$ ✓